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Jean-Yves Charbonnel

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JEAN-YVES CHARBONNEL

ABSTRACT. The commuting variety of a reductive Lie algebra $\mathfrak{g}$ is the underlying variety of a well defined subscheme of $\mathfrak{g} \times \mathfrak{g}$. In this note, it is proved that this scheme is normal. In particular, its ideal of definition is a prime ideal.

CONTENTS

1. INTRODUCTION

In this note, the base field $\Bbbk$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, ℓ is its rank, and G is its adjoint group.

1.1. The dual of $\mathfrak{g}$ identifies to $\mathfrak{g}$ by a non degenerate symmetric bilinear form on $\mathfrak{g}$ extending the Killing form of the derived algebra of $\mathfrak{g}$. Denote by $(v, w) \mapsto \langle v, w\rangle$ this bilinear form and denote by $I_{\mathfrak{g}}$ the ideal of $\Bbbk[\mathfrak{g} \times \mathfrak{g}]$ generated by the functions $(x, y) \mapsto \langle v, [x, y]\rangle$'s with v in $\mathfrak{g}$. The commuting variety $\mathcal{C}(\mathfrak{g})$ of $\mathfrak{g}$ is the subvariety of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$. It is the underlying variety to the subscheme $\mathcal{S}(\mathfrak{g})$ of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$. It is a well known and long standing open question whether or not this scheme is reduced, that is $\mathcal{C}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$. According to Richardson [Ri79], $\mathcal{C}(\mathfrak{g})$ is irreducible and according to Popov [Po08, Theorem 1], the singular locus of $\mathcal{S}(\mathfrak{g})$ has codimension at least 2 in $\mathcal{C}(\mathfrak{g})$. Then, according to Serre's normality criterion, arises the question to know whether or not $\mathcal{C}(\mathfrak{g})$ is normal. There are many results about the commuting variety. A result of Dixmier [Di79] proves that $I_{\mathfrak{g}}$ contains all the elements of the radical of $I_{\mathfrak{g}}$ which have degree 1 in the second variable. In [Ga-Gi06], Gan and Ginzburg prove that for $\mathfrak{g}$ simple of type A, the invariant elements under G of $I_{\mathfrak{g}}$ is a radical ideal of the algebra $\Bbbk[\mathfrak{g} \times \mathfrak{g}]^G$ of invariant elements of $\Bbbk[\mathfrak{g} \times \mathfrak{g}]$ under G. In [Gi12], Ginzburg proves that the normalisation of $\mathcal{C}(\mathfrak{g})$ is Cohen-Macaulay.

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1.2. Main results and sketch of proofs. According to the identfication of $\mathfrak{g}$ and its dual, $\Bbbk[\mathfrak{g}\times\mathfrak{g}]$ equals the symmetric algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$ of $\mathfrak{g}\times\mathfrak{g}$. The main result of this note is the following theorem:

Theorem 1.1. *The subscheme of* $\mathfrak{g}\times\mathfrak{g}$ *defined by* $I_{\mathfrak{g}}$ *is Cohen-Macaulay and normal. Furthermore,* $I_{\mathfrak{g}}$ *is a prime ideal of* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$.

According to Richardson's result and Popov's result, it suffices to prove that the scheme $\mathcal{S}(\mathfrak{g})$ is Cohen-Macaulay. The main idea of the proof in the theorem uses the main argument of the Dixmier's proof: for a finitely generated module M over $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$, $M=0$ if the codimension of its support is at least $l+2$ with l the projective dimension of M (see Appendix A).

Introduce the characteristic submodule of $\mathfrak{g}$, denoted by $\mathrm{B}_{\mathfrak{g}}$. By definition, $\mathrm{B}_{\mathfrak{g}}$ is a submodule of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ and an element φ of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ is in $\mathrm{B}_{\mathfrak{g}}$ if and only if for all (x,y) in $\mathfrak{g}\times\mathfrak{g}$, $\varphi(x,y)$ is in the sum of subspaces $\mathfrak{g}^{ax+by}$ with (a,b) in $\Bbbk^2\setminus\{0\}$ and $\mathfrak{g}^{ax+by}$ the centralizer of $ax+by$ in $\mathfrak{g}$. According to a Bolsinov's result, $\mathrm{B}_{\mathfrak{g}}$ is a free $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module of rank $\mathrm{b}_{\mathfrak{g}}$, the dimension of the Borel subalgebras of $\mathfrak{g}$. Moreover, the orthogonal complement of $\mathrm{B}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ is a free $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module of rank $\mathrm{b}_{\mathfrak{g}}-\ell$. These two results are fundamental in the proof of the following proposition:

Proposition 1.2. *For i positive integer, the submodule* $\bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ *of* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge^{i+\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g})$ *has projective dimension at most i.*

Denoting by E the quotient of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ by $\mathrm{B}_{\mathfrak{g}}$, let E_i be the quotient of $\bigwedge^i(E)$ by its torsion module. The $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-modules $\bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ and E_i are isomorphic. Furthermore, for $i\geq 2$, E_i is isomorphic to a direct factor of the quotient of $E\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}E_{i-1}$ by its torsion module. Denoting by $\overline{E_{i-1}}$ this quotient, the projective dimension of $\overline{E_{i-1}}$ is at most $d_{i-1}+1$ if d_{i-1} is the projective dimension of E_{i-1}, whence a proof of the proposition by induction on i.

Let d be the $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-derivation of the algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ such that for all v in $\mathfrak{g}$, $\mathrm{d}v$ is the function on $\mathfrak{g}\times\mathfrak{g}$, $(x,y)\mapsto\langle v,[x,y]\rangle$. Then the ideal of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$ generated by $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ is a graded subcomplex of the graded complex $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge(\mathfrak{g})$. The support of the homology of this complex is contained in $\mathcal{C}(\mathfrak{g})$. Then we deduce from Proposition 1.2 that this complex has no homology in degree different from $\mathrm{b}_{\mathfrak{g}}$ and $\Bbbk[\mathcal{C}(\mathfrak{g})]$ is Cohen-Macaulay by Auslander-Buchsbaum's theorem.

1.3. Notations. • For V a module over a $\Bbbk$-algebra, its symmetric and exterior algebras are denoted by $\mathrm{S}(V)$ and $\bigwedge(V)$ respectively. If E is a subset of V, the submodule of V generated by E is denoted by $\mathrm{span}(E)$. When V is a vector space over $\Bbbk$, the grassmannian of all d-dimensional subspaces of V is denoted by $\mathrm{Gr}_d(V)$.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X, denote by $\overline{Y}$ the closure of Y in X. For Y an open subset of the algebraic variety X, Y is called *a big open subset* if the codimension of $X\setminus Y$ in X is at least 2. For Y a closed subset of an algebraic variety X, its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\Bbbk[X]$ is the algebra of regular functions on X.

• All the complexes considered in this note are graded complexes over $\mathbb{Z}$ of vector spaces and their differentials are homogeneous of degree -1 and they are denoted by d. As usual, the gradation of the complex is denoted by $C_{\bullet}$.

• The dimension of the Borel subalgebras of $\mathfrak{g}$ is denoted by $\mathrm{b}_{\mathfrak{g}}$. Set $n := \mathrm{b}_{\mathfrak{g}} - \ell$ so that $\dim\mathfrak{g} = 2\mathrm{b}_{\mathfrak{g}} - \ell_{\mathfrak{g}} = 2n + \ell$.

• The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $\langle .,. \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

• For $x \in \mathfrak{g}$, denote by $\mathfrak{g}^x$ the centralizer of x in $\mathfrak{g}$. The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} \mid \dim\mathfrak{g}^x = \ell\}.$$

The subset $\mathfrak{g}_{\mathrm{reg}}$ of $\mathfrak{g}$ is a G-invariant open subset of $\mathfrak{g}$. According to [V72], $\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{reg}}$ is equidimensional of codimension 3.

• Denote by $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra of $\mathfrak{g}$-invariant elements of $\mathrm{S}(\mathfrak{g})$. Let $p_1, \dots, p_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of degree $d_1, \dots, d_\ell$ respectively. Choose the polynomials $p_1, \dots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \dots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, consider a shift of p_i in direction y: $p_i(x + ty)$ with $t \in \Bbbk$. Expanding $p_i(x + ty)$ as a polynomial in t, one obtains

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y)t^m; \qquad \forall (t, x, y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{1}$$

where $y \mapsto (m!)p_i^{(m)}(x, y)$ is the derivative at x of p_i at the order m in the direction y. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathrm{S}(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Remark that $p_i^{(0)}(x, y) = p_i(x)$ while $p_i^{(d_i)}(x, y) = p_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.3. The family $\mathcal{P}_x := \{p_i^{(m)}(x, .);\ 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $\mathrm{S}(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. One says that the family $\mathcal{P}_x$ is constructed by the *argument shift method*.

• Let $i \in \{1, \dots, \ell\}$. For x in $\mathfrak{g}$, denote by $\varepsilon_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle \varepsilon_i(x), y \rangle = \frac{\mathrm{d}}{\mathrm{d}t} p_i(x + ty)\,|_{t=0}$$

for all y in $\mathfrak{g}$. Thereby, ε_i is an invariant element of $\mathrm{S}(\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ under the canonical action of G. According to [Ko63, Theorem 9], for x in $\mathfrak{g}$, x is in $\mathfrak{g}_{\mathrm{reg}}$ if and only if $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ are linearly independent. In this case, $\varepsilon_1(x), \dots, \varepsilon_\ell(x)$ is a basis of $\mathfrak{g}^x$.

Denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ defined by the equality:

$$\varepsilon_i(x + ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x, y)t^m, \qquad \forall (t, x, y) \in \Bbbk \times \mathfrak{g} \times \mathfrak{g} \tag{2}$$

and set:

$$V_{x,y} := \mathrm{span}(\{\varepsilon_i(x, y)^{(0)}, \dots, \varepsilon_i(x, y)^{(d_i-1)},\ i = 1, \dots, \ell\})$$

for (x, y) in $\mathfrak{g} \times \mathfrak{g}$.

2. Characteristic module

For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, set:

$$V'_{x,y} = \sum_{(a,b)\in\Bbbk^2\setminus\{0\}} \mathfrak{g}^{ax+by},$$

and denote by $P_{x,y}$ the span of x and y. By definition, the characteristic module $\mathrm{B}_{\mathfrak{g}}$ of $\mathfrak{g}$ is the submodule of elements φ of $\mathrm{S}(\mathfrak{g} \times \mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ such that $\varphi(x, y)$ is in $V'_{x,y}$ for all (x, y) in $\mathfrak{g} \times \mathfrak{g}$. In this section, some properties of $\mathrm{B}_{\mathfrak{g}}$ are given.

2.1. Denote by $\Omega_{\mathfrak{g}}$ the subset of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y}$ has dimension 2 and such that $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\mathrm{reg}}$. According to [CMo08, Corollary 10], $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$.

Proposition 2.1. *Let (x, y) be in $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y} \cap \mathfrak{g}_{\mathrm{reg}}$ is not empty.*

(i) *Let O be an open subset of $\Bbbk^2$ such that $ax + by$ is in $\mathfrak{g}_{\mathrm{reg}}$ for all (a, b) in O. Then $V_{x,y}$ is the sum of the $\mathfrak{g}^{ax+by}$'s, $(a, b) \in O$.*

(ii) *The spaces $[x, V_{x,y}]$ and $[y, V_{x,y}]$ are equal.*

(iii) *The space $V_{x,y}$ has dimension at most $\mathrm{b}_{\mathfrak{g}}$ and the equality holds if and only if (x, y) is in $\Omega_{\mathfrak{g}}$.*

(iv) *The space $[x, V_{x,y}]$ is orthogonal to $V_{x,y}$. Furthermore, (x, y) is in $\Omega_{\mathfrak{g}}$ if and only if $[x, V_{x,y}]$ is the orthogonal complement of $V_{x,y}$ in $\mathfrak{g}$.*

(v) *The space $V_{x,y}$ is contained in $V'_{x,y}$. Moreover, $V_{x,y} = V'_{x,y}$ if (x, y) is in $\Omega_{\mathfrak{g}}$.*

(vi) *For $i = 1, \ldots, \ell$ and for $m = 0, \ldots, d_i - 1$, $\varepsilon_i^{(m)}$ is a G-equivariant map.*

Proof. (i) For pairwise different elements $t_{i,1}, \ldots, t_{i,d_i-1}$, $i = 1, \ldots, \ell$ of $\Bbbk \setminus \{0\}$, the $\varepsilon_i^{(m)}(x, y)$'s, $m = 0, \ldots, d_i - 1$ are linear combinations of the $\varepsilon_i(x + t_{i,j}y)$'s, $j = 1, \ldots, d_i - 1$ for $i = 1, \ldots, \ell$. Furthermore, for all z in $\mathfrak{g}_{\mathrm{reg}}$, $\varepsilon_1(z), \ldots, \varepsilon_\ell(z)$ is a basis of $\mathfrak{g}^z$ by [Ko63, Theorem 9], whence the assertion since the maps $\varepsilon_1, \ldots, \varepsilon_\ell$ are homogeneous.

(ii) Let O be an open subset of $(\Bbbk \setminus \{0\})^2$ such that $ax + by$ is in $\mathfrak{g}_{\mathrm{reg}}$ for all (a, b) in O. For all (a, b) in O, $[x, \mathfrak{g}^{ax+by}] = [y, \mathfrak{g}^{ax+by}]$ since $[ax + by, \mathfrak{g}^{ax+by}] = 0$ and since $ab \neq 0$, whence the assertion by (i).

(iii) According to [Bou02, Ch. V, §5, Proposition 3],

$$d_1 + \cdots + d_\ell = \mathrm{b}_{\mathfrak{g}}.$$

So $V_{x,y}$ has dimension at most $\mathrm{b}_{\mathfrak{g}}$. By [Bol91, Theorem 2.1], $V_{x,y}$ has dimension $\mathrm{b}_{\mathfrak{g}}$ if and only if (x, y) is in $\Omega_{\mathfrak{g}}$.

(iv) According to [Bol91, Theorem 2.1], $V_{x,y}$ is a totally isotropic subspace with respect to the skew bilinear form on $\mathfrak{g}$

$$(v, w) \longmapsto \langle ax + by, [v, w]\rangle$$

for all (a, b) in $\Bbbk^2$. As a result, by invariance of $\langle .,. \rangle$, $V_{x,y}$ is orthogonal to $[x, V_{x,y}]$. If (x, y) is in $\Omega_{\mathfrak{g}}$, $\mathfrak{g}^x$ has dimension ℓ and it is contained in $V_{x,y}$. Hence, by (iii),

$$\dim [x, V_{x,y}] = \mathrm{b}_{\mathfrak{g}} - \ell = \dim \mathfrak{g} - \dim V_{x,y}$$

so that $[x, V_{x,y}]$ is the orthogonal complement of $V_{x,y}$ in $\mathfrak{g}$. Conversely, if $[x, V_{x,y}]$ is the orhogonal complement of $V_{x,y}$ in $\mathfrak{g}$, then

$$\dim V_{x,y} + \dim [x, V_{x,y}] = \dim \mathfrak{g}.$$

Since $P_{x,y}\cap\mathfrak{g}_{\rm reg}$ is not empty, $\mathfrak{g}^{ax+by}\cap V_{x,y}$ has dimension ℓ for all (a,b) in a dense open subset of $\Bbbk^2$. By continuity, $\mathfrak{g}^x\cap V_{x,y}$ has dimension at least ℓ so that

$$2\dim V_{x,y}-\ell\geq \dim\mathfrak{g}.$$

Hence, by (iii), (x,y) is in $\Omega_{\mathfrak{g}}$.

(v) According to [Ko63, Theorem 9], for all z in $\mathfrak{g}$ and for $i=1,\dots,\ell$, $\varepsilon_i(z)$ is in $\mathfrak{g}^z$. Hence for all t in $\Bbbk$, $\varepsilon_i(x+ty)$ is in $V'_{x,y}$. So $\varepsilon_i^{(m)}(x,y)$ is in $V'_{x,y}$ for all m, whence $V_{x,y}\subset V'_{x,y}$.

Suppose that (x,y) is in $\Omega_{\mathfrak{g}}$. According to [Ko63, Theorem 9], for all (a,b) in $\Bbbk^2\setminus\{0\}$, $\varepsilon_1(ax+by),\dots,\varepsilon_\ell(ax+by)$ is a basis of $\mathfrak{g}^{ax+by}$. Hence $\mathfrak{g}^{ax+by}$ is contained in $V_{x,y}$, whence the assertion.

(vi) Let i be in $\{1,\dots,\ell\}$. Since p_i is G-invariant, ε_i is a G-equivariant map. As a result, its 2-polarizations $\varepsilon_i^{(0)},\dots,\varepsilon_i^{(d_i-1)}$ are G-equivariant under the diagonal action of G in $\mathfrak{g}\times\mathfrak{g}$. □

Theorem 2.2. (i) *The module* $\mathrm{B}_{\mathfrak{g}}$ *is a free module of rank* $\mathrm{b}_{\mathfrak{g}}$ *whose a basis is the sequence* $\varepsilon_i^{(0)},\dots,\varepsilon_i^{(d_i-1)}$, $i=1,\dots,\ell$.

(ii) *For* φ *in* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$, φ *is in* $\mathrm{B}_{\mathfrak{g}}$ *if and only if* $p\varphi\in\mathrm{B}_{\mathfrak{g}}$ *for some* p *in* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\setminus\{0\}$.

(iii) *For all* φ *in* $\mathrm{B}_{\mathfrak{g}}$ *and for all* (x,y) *in* $\mathfrak{g}\times\mathfrak{g}$, $\varphi(x,y)$ *is orthogonal to* $[x,y]$.

Proof. (i) and (ii) According to Proposition 2.1,(v), $\varepsilon_i^{(m)}$ is in $\mathrm{B}_{\mathfrak{g}}$ for all (i,m). Moreover, according to Proposition 2.1,(iii), these elements are linearly independent over $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$. Let φ be an element of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ such that $p\varphi$ is in $\mathrm{B}_{\mathfrak{g}}$ for some p in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\setminus\{0\}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g}\times\mathfrak{g}$, for all (x,y) in a dense open subset of $\Omega_{\mathfrak{g}}$, $\varphi(x,y)$ is in $V_{x,y}$ by Proposition 2.1,(v). According to Proposition 2.1,(iii), the map

$$\Omega_{\mathfrak{g}}\longrightarrow \mathrm{Gr}_{\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g}),\qquad (x,y)\longmapsto V_{x,y}$$

is regular. So, $\varphi(x,y)$ is in $V_{x,y}$ for all (x,y) in $\Omega_{\mathfrak{g}}$ and for some regular functions $a_{i,m}$, $i=1,\dots,\ell$, $m=0,\dots,d_i-1$ on $\Omega_{\mathfrak{g}}$,

$$\varphi(x,y)=\sum_{i=1}^{\ell}\sum_{m=0}^{d_i-1}a_{i,m}(x,y)\varepsilon_i^{(m)}(x,y)$$

for all (x,y) in $\Omega_{\mathfrak{g}}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g}\times\mathfrak{g}$ and since $\mathfrak{g}\times\mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g}\times\mathfrak{g}$. Hence φ is a linear combination of the $\varepsilon_i^{(m)}$'s with coefficients in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$. As a result, the sequence $\varepsilon_i^{(m)}$, $i=1,\dots,\ell$, $m=0,\dots,d_i-1$ is a basis of the module $\mathrm{B}_{\mathfrak{g}}$ and $\mathrm{B}_{\mathfrak{g}}$ is the subset of elements φ of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ such that $p\varphi\in\mathrm{B}_{\mathfrak{g}}$ for some p in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\setminus\{0\}$.

(iii) Let φ be in $\mathrm{B}_{\mathfrak{g}}$. According to (i) and Proposition 2.1,(iv), for all (x,y) in $\Omega_{\mathfrak{g}}$, $[x,\varphi(x,y)]$ is orthogonal to $V_{x,y}$. Then, since y is in $V_{x,y}$, $[x,\varphi(x,y)]$ is orthogonal to y and $\langle\varphi(x,y),[x,y]\rangle=0$, whence the assertion. □

2.2. Also denote by $\langle .,.\rangle$ the natural extension of $\langle .,.\rangle$ to the module $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$.

Proposition 2.3. *Let* $\mathrm{C}_{\mathfrak{g}}$ *be the orthogonal complement of* $\mathrm{B}_{\mathfrak{g}}$ *in* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$.

(i) *For* φ *in* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$, φ *is in* $\mathrm{C}_{\mathfrak{g}}$ *if and only if* $\varphi(x,y)$ *is in* $[x,V_{x,y}]$ *for all* (x,y) *in a nonempty open subset of* $\mathfrak{g}\times\mathfrak{g}$.

(ii) *The module* $\mathrm{C}_{\mathfrak{g}}$ *is free of rank* $\mathrm{b}_{\mathfrak{g}}-\ell$. *Furthermore, the sequence of maps*

$$(x,y)\mapsto[x,\varepsilon_i^{(1)}(x,y)],\dots,(x,y)\mapsto[x,\varepsilon_i^{(d_i-1)}(x,y)],\ i=1,\dots,\ell$$

is a basis of $\mathrm{C}_{\mathfrak{g}}$.

(iii) *The orthogonal complement of* $\mathrm{C}_{\mathfrak{g}}$ *in* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ *equals* $\mathrm{B}_{\mathfrak{g}}$.

Proof. (i) Let φ be in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$. If φ is in $\mathrm{C}_{\mathfrak{g}}$, then $\varphi(x,y)$ is orthogonal to $V_{x,y}$ for all (x,y) in $\Omega_{\mathfrak{g}}$. Then, according to Proposition 2.1,(iv), $\varphi(x,y)$ is in $[x,V_{x,y}]$ for all (x,y) in $\Omega_{\mathfrak{g}}$. Conversely, suppose that $\varphi(x,y)$ is in $[x,V_{x,y}]$ for all (x,y) in a nonempty open subset V of $\mathfrak{g}\times\mathfrak{g}$. By Proposition 2.1,(iv) again, for all (x,y) in $V\cap\Omega_{\mathfrak{g}}$, $\varphi(x,y)$ is orthogonal to the $\varepsilon_i^{(m)}(x,y)$'s, $i=1,\ldots,\ell$, $m=0,\ldots,d_i-1$, whence the assertion by Theorem 2.1.

(ii) Let C be the submodule of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ generated by the maps

$$(x,y)\mapsto[x,\varepsilon_i^{(1)}(x,y)],\ldots,(x,y)\mapsto[x,\varepsilon_i^{(d_i-1)}(x,y)],\ i=1,\ldots,\ell$$

According to (i), C is a submodule of $\mathrm{C}_{\mathfrak{g}}$. This module is free of rank $\mathrm{b}_{\mathfrak{g}}-\ell$ since $[x,V_{x,y}]$ has dimension $\mathrm{b}_{\mathfrak{g}}-\ell$ for all (x,y) in $\Omega_{\mathfrak{g}}$ by Proposition 2.1, (iii) and (iv). According to (i), for φ in $\mathrm{C}_{\mathfrak{g}}$, for all (x,y) in $\Omega_{\mathfrak{g}}$,

$$\varphi(x,y)=\sum_{i=1}^{\ell}\sum_{m=1}^{d_i-1}a_{i,m}(x,y)[x,\varepsilon_i^{(m)}(x,y)]$$

with the $a_{i,m}$'s regular on $\Omega_{\mathfrak{g}}$ and uniquely defined by this equality. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g}\times\mathfrak{g}$ and since $\mathfrak{g}\times\mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g}\times\mathfrak{g}$. As a result, φ is in C, whence the assertion.

(iii) Let φ be in the orthogonal complement of $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$. According to (ii), for all (x,y) in $\Omega_{\mathfrak{g}}$, $\varphi(x,y)$ is orthogonal to $[x,V_{x,y}]$. Hence by Proposition 2.1,(iv), $\varphi(x,y)$ is in $V_{x,y}$ for all (x,y) in $\Omega_{\mathfrak{g}}$. So, by Theorem 2.1, φ is in $\mathrm{B}_{\mathfrak{g}}$, whence the assertion. □

Denote by $\mathcal{B}$ and $\mathcal{C}$ the localizations of $\mathrm{B}_{\mathfrak{g}}$ and $\mathrm{C}_{\mathfrak{g}}$ on $\mathfrak{g}\times\mathfrak{g}$ respectively. For (x,y) in $\mathfrak{g}\times\mathfrak{g}$, let $C_{x,y}$ be the image of $\mathrm{C}_{\mathfrak{g}}$ by the evaluation map at (x,y).

Lemma 2.4. *There exists an affine open cover* $\mathcal{O}$ *of* $\Omega_{\mathfrak{g}}$ *verifying the following condition: for all O in* $\mathcal{O}$*, there exist some subspaces E and F of* $\mathfrak{g}$*, depending on O, such that*

$$\mathfrak{g}=E\oplus V_{x,y}=F\oplus C_{x,y}$$

for all (x,y) *in O. Moreover, for all* (x,y) *in O, the orthogonal complement of* $V_{x,y}$ *in* $\mathfrak{g}$ *equals* $C_{x,y}$.

Proof. According to Proposition 2.1,(iii) and (iv), for all (x,y) in $\Omega_{\mathfrak{g}}$, $V_{x,y}$ and $C_{x,y}$ have dimension $\mathrm{b}_{\mathfrak{g}}$ and $\mathrm{b}_{\mathfrak{g}}-\ell$ respectively so that the maps

$$\Omega_{\mathfrak{g}}\longrightarrow\mathrm{Gr}_{\mathrm{b}_{\mathfrak{g}}}(\mathfrak{g}),\qquad (x,y)\longmapsto V_{x,y},\qquad \Omega_{\mathfrak{g}}\longrightarrow\mathrm{Gr}_{\mathrm{b}_{\mathfrak{g}}-\ell}(\mathfrak{g}),\qquad (x,y)\longmapsto C_{x,y}$$

are regular, whence the assertion. □

3. Torsion and projective dimension

Let E and $E^{\#}$ be the quotients of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ by $\mathrm{B}_{\mathfrak{g}}$ and $\mathrm{C}_{\mathfrak{g}}$ respectively. For i positive integer, denote by E_i the quotient of $\bigwedge^i(E)$ by its torsion module.

3.1. Let $\mathrm{B}_{\mathfrak{g}}^*$ and $\mathrm{C}_{\mathfrak{g}}^*$ be the duals of $\mathrm{B}_{\mathfrak{g}}$ and $\mathrm{C}_{\mathfrak{g}}$ respectively.

Lemma 3.1. (i) *The* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$*-modules E and $E^{\#}$ have projective dimension at most* 1.

(ii) *The* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$*-modules E and $E^{\#}$ are torsion free.*

(iii) *The modules* $\mathrm{C}_{\mathfrak{g}}$ *and* $\mathrm{B}_{\mathfrak{g}}$ *are the duals of E and $E^{\#}$ respectively.*

(iv) *The canonical morphism from E to* $\mathrm{C}_{\mathfrak{g}}^*$ *is an embedding.*

Proof. (i) By definition, the short sequences of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-modules,

$$0 \longrightarrow \mathrm{B}_{\mathfrak{g}} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g} \longrightarrow E \longrightarrow 0$$

$$0 \longrightarrow \mathrm{C}_{\mathfrak{g}} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g} \longrightarrow E^{\#} \longrightarrow 0$$

are exact. Hence E and $E^{\#}$ have projective dimension at most 1 since $\mathrm{B}_{\mathfrak{g}}$ and $\mathrm{C}_{\mathfrak{g}}$ are free modules by Theorem 2.2 and Proposition 2.3,(ii).

(ii) The module E is torsion free by Theorem 2.2,(ii). By definition, for φ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$, φ is in $\mathrm{C}_{\mathfrak{g}}$ if $p\varphi$ is in $\mathrm{C}_{\mathfrak{g}}$ for some p in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\setminus\{0\}$, whence $E^{\#}$ is torsion free.

(iii) According to the exact sequences of (i), the dual of E is the orthogonal complement of $\mathrm{B}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ and the dual of $E^{\#}$ is the orthogonal complement of $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$, whence the assertion since $\mathrm{C}_{\mathfrak{g}}$ is the orthogonal complement of $\mathrm{B}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ by definition and since $\mathrm{B}_{\mathfrak{g}}$ is the orthogonal complement of $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ by Proposition 2.3,(iii).

(iv) Let $\overline{\omega}$ be in the kernel of the canonical morphism from E to $\mathrm{C}_{\mathfrak{g}}^*$. Let ω be a representative of $\overline{\omega}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$. According to Proposition 2.3,(iii), $\mathrm{B}_{\mathfrak{g}}$ is the orthogonal complement of $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g}$ so that ω is in $\mathrm{B}_{\mathfrak{g}}$, whence the assertion. □

Set:

$$\varepsilon = \wedge_{i=1}^{\ell}\varepsilon_i^{(0)}\wedge\cdots\wedge\varepsilon_i^{(d_i-1)}$$

and for i positive integer, denote by θ_i the morphism

$$\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\textstyle\bigwedge^i(\mathfrak{g}) \longrightarrow \bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}), \qquad \varphi\longmapsto\varphi\wedge\varepsilon.$$

Proposition 3.2. *Let i be a positive integer.*

(i) *The morphism θ_i defines through the quotient an isomorphism from E_i onto* $\bigwedge^i(\mathfrak{g})\wedge\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$.

(ii) *The short sequence of* $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$*-modules*

$$0 \longrightarrow \mathrm{B}_{\mathfrak{g}}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}E_i \longrightarrow \mathfrak{g}\otimes_{\Bbbk}E_i \longrightarrow E\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}E_i \longrightarrow 0$$

is exact.

Proof. (i) For j positive integer, denote by π_j the canonical map from $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\bigwedge^j(\mathfrak{g})$ to $\bigwedge^j(E)$. Let ω be in the kernel of π_i. Let O be an element of the affine open cover of $\Omega_{\mathfrak{g}}$ of Lemma 2.4 and let W be a subspace of $\mathfrak{g}$ such that

$$\mathfrak{g} = W\oplus V_{x,y}$$

for all (x,y) in O so that π_1 induces an isomorphism

$$\Bbbk[O]\otimes_{\Bbbk}W \longrightarrow \Bbbk[O]\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}E$$

Moreover, $\mathrm{B}_{\mathfrak{g}}$ is the kernel of π_1. Then, from the equality

$$\Bbbk[O] \otimes_{\Bbbk} \textstyle\bigwedge^i(\mathfrak{g}) = \displaystyle\bigoplus_{j=0}^{i} \textstyle\bigwedge^j(W) \wedge \Bbbk[O] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \textstyle\bigwedge^{i-j}(\mathrm{B}_{\mathfrak{g}})$$

it results that the restriction of ω to O is in $\Bbbk[O] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^{i-1}(\mathfrak{g}) \wedge \mathrm{B}_{\mathfrak{g}}$. Hence the restriction of $\omega \wedge \varepsilon$ to O equals 0 and ω is in the kernel of θ_i since $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ has no torsion as a submodule of a free module. As a result, θ_i defines through the quotient a morphism from $\bigwedge^i(E)$ to $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$. Denote it by ϑ'_i. Since $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ is torsion free, the torsion submodule of $\bigwedge^i(E)$ is contained in the kernel of ϑ'_i. Hence ϑ'_i defines through the quotient a morphism from E_i to $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$. Denoting it by ϑ_i, ϑ'_i and ϑ_i are surjective since too is θ_i.

Let $\overline{\omega}$ be in the kernel of ϑ'_i and let ω be a representative of $\overline{\omega}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \bigwedge^i(\mathfrak{g})$. Then $\omega \wedge \varepsilon = 0$ so that the restriction of ω to the above open subset O is in $\Bbbk[O] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^{i-1}(\mathfrak{g}) \wedge \mathrm{B}_{\mathfrak{g}}$. As a result, the restriction of $\overline{\omega}$ to O equals 0. So, $\overline{\omega}$ is in the torsion submodule of $\bigwedge^i(E)$, whence the assertion.

(ii) By definition, the sequence

$$0 \longrightarrow \mathrm{B}_{\mathfrak{g}} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g} \longrightarrow E \longrightarrow 0$$

is exact. Then the sequence

$$\mathrm{Tor}_1^{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E, E_i) \longrightarrow \mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_{\Bbbk} E_i \longrightarrow E \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow 0$$

is exact. By definition, E_i is torsion free. As a result, $\mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$ is torsion free since $\mathrm{B}_{\mathfrak{g}}$ is a free module. Then, since $\mathrm{Tor}_1^{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E, E_i)$ is a torsion module, its image in $\mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$ equals 0, whence the assertion. □

3.2. For i positive integer, $\langle .,. \rangle$ has a canonical extension to $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \bigwedge^i(\mathfrak{g})$ denoted again by $\langle .,. \rangle$.

Lemma 3.3. *Let i be a positive integer. Let T_i be the torsion module of $E \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$ and let T'_i be its inverse image by the canonical morphism $\mathfrak{g} \otimes_{\Bbbk} E_i \rightarrow E \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$.*

(i) *The canonical morphism from $\bigwedge^i(E)$ to $\bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$ defines through the quotient an embedding of E_i into $\bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$.*

(ii) *The module of T'_i is the intersection of $\mathfrak{g} \otimes_{\Bbbk} E_i$ and $\mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$.*

(iii) *The module T'_i is isomorphic to $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, E_i)$.*

Proof. (i) According to Lemma 3.1,(iii), there is a canonical morphism from $\bigwedge^i(E)$ to $\bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$. Let $\overline{\omega}$ be in its kernel and let ω be a representative of $\overline{\omega}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \bigwedge^i(\mathfrak{g})$. Then ω is orthogonal to $\bigwedge^i(\mathrm{C}_{\mathfrak{g}})$ with respect to $\langle .,. \rangle$. So for O as in Lemma 2.4, the restriction of ω to O is in $\Bbbk[O] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^{i-1}(\mathfrak{g}) \wedge \mathrm{B}_{\mathfrak{g}}$. Hence the restriction of $\overline{\omega}$ to O equals 0. In other words, $\overline{\omega}$ is in the torsion module of $\bigwedge^i(E)$, whence the assertion since $\bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$ is a free module.

(ii) Since $\bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$ is a free module, by Proposition 3.2,(ii), there is a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i & \longrightarrow & \mathfrak{g} \otimes_{\Bbbk} E_i & \longrightarrow & E \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*) & \longrightarrow & \mathfrak{g} \otimes_{\Bbbk} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*) & \longrightarrow & E \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*) & \longrightarrow & 0
\end{array}$$

Moreover, the two first vertical arrows are embeddings. Hence T'_i is the intersection of $\mathfrak{g} \otimes_{\Bbbk} E_i$ and $\mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$ in $\mathfrak{g} \otimes_{\Bbbk} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$.

(iii) According to the identification of $\mathfrak{g}$ with its dual, $\mathfrak{g} \otimes_{\Bbbk} E_i = \mathrm{Hom}_{\Bbbk}(\mathfrak{g}, E_i)$. Moreover, according to the short exact sequence of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-modules

$$0 \longrightarrow \mathrm{C}_{\mathfrak{g}} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g} \longrightarrow E^{\#} \longrightarrow 0$$

the sequence of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-modules

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, E_i) \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathfrak{g}, E_i) \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_{\mathfrak{g}}, E_i) \longrightarrow \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, E_i)$$

is exact. For φ in $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}, E_i)$, φ is in the kernel of the third arrow if and only if $\mathrm{C}_{\mathfrak{g}}$ is contained in the kernel of φ. On the other hand, according to the identification of $\mathfrak{g}$ and its dual, $\mathrm{Hom}_{\Bbbk}(\mathfrak{g}, \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$ identifies with $\mathfrak{g} \otimes_{\Bbbk} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$. By Proposition 2.3, $\mathrm{B}_{\mathfrak{g}}$ is the orthogonal complement of $\mathrm{C}_{\mathfrak{g}}$ in $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}$ and $\mathrm{C}_{\mathfrak{g}}^*$ is a free $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module. So, for ψ in $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g}, \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*))$, ψ equals 0 on $\mathrm{C}_{\mathfrak{g}}$ if and only if it is in $\mathrm{B}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bigwedge^i(\mathrm{C}_{\mathfrak{g}}^*)$, whence the assertion by (ii). □

The following corollary results from Lemma 3.3.

Corollary 3.4. *Let i be a positive integer and let $\overline{E_i}$ be the quotient of $E \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$ by its torsion module. Then the short sequence of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-modules*

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, E_i) \longrightarrow \mathfrak{g} \otimes_{\Bbbk} E_i \longrightarrow \overline{E_i} \longrightarrow 0$$

is exact.

3.3. Denote by $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}$ the category of finite $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-modules. Let ι be the morphism

$$\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g} \longrightarrow \mathrm{C}_{\mathfrak{g}}^*, \qquad v \longmapsto (\mu \mapsto \langle v, \mu\rangle)$$

Lemma 3.5. *Let $\mathrm{A}_{\mathfrak{g}}$ be the quotient of $\mathrm{C}_{\mathfrak{g}}^*$ by $\iota(\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g})$. Then the two functors $\mathrm{A}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bullet$ and $\mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, \bullet)$ of the category $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}$ are isomorphic.*

Proof. For d nonnegative integer, denote by $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(d)$ the full subcategory of $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}$ whose objects are the modules of projective dimension at most d. Prove by induction on d that the restrictions to $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(d)$ of the functors $\mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, \bullet)$ and $\mathrm{A}_{\mathfrak{g}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bullet$ are isomorphic. Let M be a finite $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module. Denoting by d its projective dimension, there is a short exact sequence

$$0 \longrightarrow Z \longrightarrow P \longrightarrow M \longrightarrow 0$$

with Z a module of projective dimension $d-1$ if $d > 0$ and $Z = 0$ otherwise.

Suppose $d = 0$. Then, from the short exact sequence

$$0 \longrightarrow \mathrm{C}_{\mathfrak{g}} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_{\Bbbk} \mathfrak{g} \longrightarrow E^{\#} \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, M) \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathfrak{g}, M) \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_{\mathfrak{g}}, M) \longrightarrow \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, M) \longrightarrow 0.$$

Since $\mathrm{C}_{\mathfrak{g}}$ is a free module, $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_{\mathfrak{g}}, M)$ is functorially isomorphic to $\mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M$. Then by the right exactness of the functor $\mathrm{A}_{\mathfrak{g}}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}\bullet$, there is an isomorphism of exact sequences

$$\begin{tikzcd}
\mathrm{Hom}_{\Bbbk}(\mathfrak{g}, M) \arrow[r] & \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_{\mathfrak{g}}, M) \arrow[r] & \mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, M) \arrow[r] & 0 \\
\mathfrak{g}\otimes_{\Bbbk} M \arrow[r] \arrow[u, "\delta_0"] & \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M \arrow[r] \arrow[u, "\delta_1"] & \mathrm{A}_{\mathfrak{g}}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

Since the two sequences depends functorially on M, from the isomorphisms of functors

$$\mathfrak{g}\otimes_{\Bbbk}\bullet \longrightarrow \mathrm{Hom}_{\Bbbk}(\mathfrak{g},\bullet), \quad \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}\bullet \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_{\mathfrak{g}},\bullet),$$

we deduce that the restrictions to $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(0)$ of the functors $\mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#},\bullet)$ and $\mathrm{A}_{\mathfrak{g}}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}\bullet$ are isomorphic.

Suppose the statement true for $d-1$. Setting $Q := \iota(\mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g})$, one has two short exact sequences

$$0 \longrightarrow \mathrm{B}_{\mathfrak{g}} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g})\otimes_{\Bbbk}\mathfrak{g} \longrightarrow Q \longrightarrow 0, \qquad 0 \longrightarrow Q \longrightarrow \mathrm{C}_{\mathfrak{g}}^{*} \longrightarrow \mathrm{A}_{\mathfrak{g}} \longrightarrow 0.$$

Since $E^{\#}$ has projective dimension at most 1 by Lemma 3.1, $\mathrm{Ext}^{2}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, Z) = 0$. Then, by induction hypothesis, one has a commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & \mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, Z) \arrow[r, "\mathrm{d}"] \arrow[u] & \mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, P) \arrow[r, "\mathrm{d}"] \arrow[u] & \mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, M) \arrow[r] \arrow[u, dotted] & 0 \\
0 \arrow[r] & \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} Z \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M \arrow[r] \arrow[u, dotted, "\delta_M"] & 0 \\
 & Q\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} Z \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & Q\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P \arrow[r, "\mathrm{d}"] \arrow[u, "\delta"] & Q\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

with exact lines and columns since $\mathrm{C}_{\mathfrak{g}}^{*}$ is a free module. Let a and a' be in $\mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P$ such that $\mathrm{d}a = \mathrm{d}a'$. Then $a - a' = \mathrm{d}a_1$ with a_1 in $\mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} Z$ so that

$$\mathrm{d}\circ\delta a - \mathrm{d}\circ\delta a' = \mathrm{d}\circ\delta\circ\mathrm{d}a_1 = 0$$

whence a morphism

$$\mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M \xrightarrow{\delta_M} \mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, M)$$

uniquely defined by the equality $\delta_M\circ\mathrm{d} = \mathrm{d}\circ\delta$.

Let a be in $\mathrm{Ext}^{1}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^{\#}, M)$. Then

$$a = \mathrm{d}\circ\delta a_1 = \delta_M\circ\mathrm{d}a_1 \quad \text{with} \quad a_1 \in \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P.$$

Hence δ_M is surjective. Let b be in the kernel of δ_M. Then

$$b = \mathrm{d}b_1, \quad \mathrm{d}\circ\delta b_1 = 0, \quad \delta b_1 = \mathrm{d}\circ\delta b_2 \quad \text{with} \quad b_1 \in \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P, \quad b_2 \in \mathrm{C}_{\mathfrak{g}}^{*}\otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} Z.$$

so that $b_1 - \mathrm{d}b_2 = \delta b_3$ with b_3 in $Q \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P$, whence $b = \delta\circ\mathrm{d}b_3$. As a result, the above diagram is canonically completed by an exact third column and one has an isomorphism of short exact sequences

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, Z) \arrow[r] & \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, P) \arrow[r] & \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, M) \arrow[r] & 0 \\ \mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} Z \arrow[r] \arrow[u] & \mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} P \arrow[r] \arrow[u] & \mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} M \arrow[r] \arrow[u] & 0 \end{tikzcd} .$$

Since the two sequences depends functorially on the short exact sequence

$$0 \longrightarrow Z \longrightarrow P \longrightarrow M \longrightarrow 0$$

and since the restrictions to $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(d-1)$ of the two functors $\mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, \bullet)$ and $\mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \bullet$ are isomorphic, too is their restrictions to $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(d)$, whence the lemma since all object of $\mathrm{Mod}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}$ has a finite projective dimension. □

From the exact sequence,

$$0 \longrightarrow \mathrm{B}_\mathfrak{g} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \mathfrak{g} \longrightarrow \mathrm{C}^*_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{g} \longrightarrow 0,$$

we deduce the graded homology complex,

$$0 \longrightarrow \mathrm{B}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_\Bbbk E_i \longrightarrow \mathrm{C}^*_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow \mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow 0$$

denoted by $C_\bullet$. For i positive integer, let d_i and d'_i be the projective dimensions of E_i and $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$.

Lemma 3.6. *Let Q be the space of cycles of degree 2 of the complex $C_\bullet$.*
(i) *Denoting by d''_i the projective dimension of* $\mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$*, d'_i is at most* $\sup\{d''_i - 2, d_i\}$.
(ii) *The complex $C_\bullet$ has no homology in degree* 0, 1 *and* 3. *Moreover, Q identifies with* $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$.
(iii) *The module* $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$ *has projective dimension at most d_i.*

Proof. (i) From the short exact sequence

$$0 \longrightarrow \mathrm{C}_\mathfrak{g} \longrightarrow \mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \mathfrak{g} \longrightarrow E^\# \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i) \longrightarrow \mathrm{Hom}_\Bbbk(\mathfrak{g}, E_i) \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_\mathfrak{g}, E_i) \longrightarrow \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i) \longrightarrow 0$$

whence the two short exact sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i) \longrightarrow \mathrm{Hom}_\Bbbk(\mathfrak{g}, E_i) \longrightarrow Z \longrightarrow 0$$

$$0 \longrightarrow Z \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_\mathfrak{g}, E_i) \longrightarrow \mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i) \longrightarrow 0$$

with Z the image of the arrow

$$\mathrm{Hom}_\Bbbk(\mathfrak{g}, E_i) \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(\mathrm{C}_\mathfrak{g}, E_i)$$

Denoting by d the projective dimension of Z, one deduces the inequalities

$$d'_i \leq \sup\{d-1, d_i\}, \qquad d \leq \sup\{d''_i - 1, d_i\}$$

since $\mathrm{C}_\mathfrak{g}$ is a free module, whence the assertion.

(ii) By right exactness of the functor $\bullet \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$, $C_\bullet$ has no homology in degree 0 and 1. Moreover, its space of cycles of degree 3 is a torsion submodule of C_3. Since E_i is torsion free and since $\mathrm{B}_\mathfrak{g}$ is free, C_3 has no torsion. Hence $C_\bullet$ has no homology in degree 3. According to Lemma 3.3,(ii) and (iii), $\mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$ identifies with a submodule of $\mathfrak{g} \otimes_\Bbbk E_i$. According to these identifications, Q is the space of morphisms from $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \mathfrak{g}$ to E_i, equal to 0 on $\mathrm{C}_\mathfrak{g}$, that is $Q = \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$.

(iii) By (ii), one has a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i) \longrightarrow \mathrm{C}_\mathfrak{g}^* \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow \mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i \longrightarrow 0.$$

So, $\mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$ has projective dimension at most $\sup\{d'_i + 1, d_i\}$. According to Lemma 3.5, $\mathrm{A}_\mathfrak{g} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} E_i$ and $\mathrm{Ext}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(E^\#, E_i)$ are isomorphic. So by (i),

$$d'_i \leq \sup\{d'_i - 1, d_i\},$$

whence $d'_i \leq d_i$. □

The following corollary results from Corollary 3.4 and Lemma 3.6,(iii), since $\mathrm{B}_\mathfrak{g}$ is free.

Corollary 3.7. *Let i be a positive integer. Then $\overline{E_i}$ has projective dimension at most $d_i + 1$.*

3.4. For i a positive integer and for M a $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-module, let consider on $M^{\otimes i}$ the canonical action of the symmetric group $\mathfrak{S}_i$. For σ in $\mathfrak{S}_i$, denote by $\epsilon(\sigma)$ its signature. Let $M^{\otimes i}_{\mathrm{sign}}$ be the submodule of elements a of $M^{\otimes i}$ such that $\sigma.a = \epsilon(\sigma)a$ for all σ in $\mathfrak{S}_i$ and let δ_i be the endomorphism of $M^{\otimes i}$,

$$a \longmapsto \delta_i(a) = \frac{1}{i!}\sum_{\sigma\in\mathfrak{S}_i} \epsilon(\sigma)\sigma.a.$$

Then δ_i is a projection of $M^{\otimes i}$ onto $M^{\otimes i}_{\mathrm{sign}}$.

For L submodule of $\mathrm{C}_\mathfrak{g}^*$, denote by L_i the image of $L^{\otimes i}$ by the canonical map from $L^{\otimes i}$ to $(\mathrm{C}_\mathfrak{g}^*)^{\otimes i}$ and set $L_{i,\mathrm{sign}} := L_i \cap (\mathrm{C}_\mathfrak{g}^*)^{\otimes i}_{\mathrm{sign}}$. Let $\overline{\bigwedge^i(L)}$ be the quotient of $\bigwedge^i(L)$ by its torsion module. For $i \geq 2$, identify $\mathfrak{S}_{i-1}$ with the stabilizer of i in $\mathfrak{S}_i$ and denote by $L_{i-1,\mathrm{sign},1}$ the submodule of elements a of L_i such that $\sigma.a = \epsilon(\sigma)a$ for all σ in $\mathfrak{S}_{i-1}$.

Lemma 3.8. *Let i be a positive integer and let L be a submodule of* $\mathrm{C}_\mathfrak{g}^*$.

(i) *The module L_i is isomorphic to the quotient of $L^{\otimes i}$ by its torsion module.*

(ii) *The module $L_{i,\mathrm{sign}}$ is isomorphic to $\overline{\bigwedge^i(L)}$.*

(iii) *For $i \geq 2$, the module $L_{i,\mathrm{sign}}$ is a direct factor of $L_{i-1,\mathrm{sign},1}$.*

(iv) *For $i \geq 2$, the module $L_{i-1,\mathrm{sign},1}$ is isomorphic to the quotient of $\overline{\bigwedge^{i-1}(L)} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L$ by its torsion module.*

Proof. (i) Let L_1 and L_2 be submodules of a free module F over $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$. From the short exact sequence

$$0 \longrightarrow L_2 \longrightarrow F \longrightarrow F/L_2 \longrightarrow 0$$

one deduces the exact sequence

$$\mathrm{Tor}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(L_1, L_2) \longrightarrow L_1 \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L_2 \longrightarrow L_1 \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} F \longrightarrow L_1 \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} (F/L_2) \longrightarrow 0.$$

Since F is free, $L_1 \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} F$ is torsion free. Hence the kernel of the second arrow is the torsion module of $L_1 \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L_2$ since $\mathrm{Tor}^1_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})}(L_1, L_2)$ is a torsion module, whence the assertion by induction on i.

(ii) There is a commutative diagram

$$\begin{tikzcd} L^{\otimes i} \arrow[r] \arrow[d, "\delta_i"'] & (\mathrm{C}^*_{\mathfrak{g}})^{\otimes i} \arrow[d, "\delta_i"] \\ L^{\otimes i}_{\mathrm{sign}} \arrow[r] & (\mathrm{C}^*_{\mathfrak{g}})^{\otimes i}_{\mathrm{sign}} \end{tikzcd}$$

so that $L_{i,\mathrm{sign}}$ is the image of $L^{\otimes i}_{\mathrm{sign}}$ by the canonical morphism $L^{\otimes i} \longrightarrow (\mathrm{C}^*_{\mathfrak{g}})^{\otimes i}$, whence a commutative diagram

$$\begin{tikzcd} L^{\otimes i}_{\mathrm{sign}} \arrow[r] \arrow[d] & \bigwedge^i(L) \arrow[d] \\ (\mathrm{C}^*_{\mathfrak{g}})^{\otimes i}_{\mathrm{sign}} \arrow[r] & \bigwedge^i(\mathrm{C}^*_{\mathfrak{g}}) \end{tikzcd}$$

According to (i), the kernel of the left down arrow is the torsion module of $L^{\otimes i}_{\mathrm{sign}}$ so that the kernel of the right down arrow is the torsion module of $\bigwedge^i(L)$ since the horizontal arrows are isomorphisms. Moreover, the image of $L_{i,\mathrm{sign}}$ in $\bigwedge^i(\mathrm{C}^*_{\mathfrak{g}})$ is the image of $\bigwedge^i(L)$. Hence $\overline{\bigwedge^i(L)}$ is isomorphic to $L_{i,\mathrm{sign}}$.

(iii) Denote by Q_i the kernel of the endomorphism δ_i of $(\mathrm{C}^*_{\mathfrak{g}})^{\otimes i}$. Since δ_i is a projection onto $(\mathrm{C}^*_{\mathfrak{g}})^{\otimes i}_{\mathrm{sign}}$ such that $\delta_i(L_i)$ is contained in $L_{i,\mathrm{sign}}$,

$$(\mathrm{C}^*_{\mathfrak{g}})^{\otimes i} = (\mathrm{C}^*_{\mathfrak{g}})^{\otimes i}_{\mathrm{sign}} \oplus Q_i, \qquad L_i = L_{i,\mathrm{sign}} \oplus Q_i \cap L_i$$

whence

$$L_{i-1,\mathrm{sign},1} = L_{i,\mathrm{sgn}} \oplus Q_i \cap L_{i-1,\mathrm{sign},1}$$

since $L_{i,\mathrm{sign}}$ is a submodule of $L_{i-1,\mathrm{sign},1}$.

(iv) Let L'_i be the image of $L_{i-1,\mathrm{sign},1}$ by the canonical morphism $(\mathrm{C}^*_{\mathfrak{g}})^{\otimes i} \to \bigwedge^{i-1}(\mathrm{C}^*_{\mathfrak{g}}) \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{C}^*_{\mathfrak{g}}$. Then L'_i is contained in $\bigwedge^{i-1}(\mathrm{C}^*_{\mathfrak{g}}) \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L$ since $\bigwedge^{i-1}(\mathrm{C}^*_{\mathfrak{g}}) \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L$ is a submodule of $\bigwedge^{i-1}(\mathrm{C}^*_{\mathfrak{g}}) \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{C}^*_{\mathfrak{g}}$. Moreover, the morphism $L_{i-1,\mathrm{sign},1} \to L'_i$ is an isomorphism since too is the morphism

$$(\mathrm{C}^*_{\mathfrak{g}})^{\otimes(i-1)}_{\mathrm{sign}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{C}^*_{\mathfrak{g}} \longrightarrow \bigwedge^{i-1}(\mathrm{C}^*_{\mathfrak{g}}) \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} \mathrm{C}^*_{\mathfrak{g}}.$$

From (ii), it results the commutative diagram

$$\begin{tikzcd} L^{\otimes(i-1)}_{\mathrm{sign}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L \arrow[r] \arrow[d] & \overline{\bigwedge^{i-1}(L)} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L \arrow[d] \\ L_{i-1,\mathrm{sign},1} \arrow[r] & L'_i \end{tikzcd}$$

with the right down arrow surjective. According to (i), the kernel of the left down arrow is the torsion module of $L^{\otimes(i-1)}_{\mathrm{sign}} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L$. Hence the kernel of the right down arrow is the torsion module of $\overline{\bigwedge^{i-1}(L)} \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} L$, whence the assertion. $\square$

Proposition 3.9. *Let i be a positive integer. Then E_i and $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$ have projective dimension at most i.*

Proof. According to Proposition 3.2,(i), the modules E_i and $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$ are isomorphic. Prove by induction on i that E_i has projective dimension at most i. By Lemma 3.1,(i), it is true for $i = 1$. Suppose that it is true for $i-1$. According to Corollary 3.7, $\overline{E_{i-1}}$ has projective dimension at most i. By Lemma 3.8, for $L = E$, E_i is a direct factor of $\overline{E_{i-1}}$ since E is a submodule of $\mathrm{C}_\mathfrak{g}^*$ by Lemma 3.1,(iv) and since $E_i = \overline{\bigwedge^i(E)}$. Hence E_i has projective dimension at most i. □

4. Main results

Let $I_\mathfrak{g}$ be the ideal of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$ generated by the functions $(x,y) \mapsto \langle v, [x,y]\rangle$ with v in $\mathfrak{g}$. The nullvariety of $I_\mathfrak{g}$ in $\mathfrak{g}\times\mathfrak{g}$ is $\mathcal{C}(\mathfrak{g})$. Let d be the $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})$-derivation of the algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$ such that $\mathrm{d}v$ is the function $(x,y) \mapsto \langle v,[x,y]\rangle$ on $\mathfrak{g}\times\mathfrak{g}$ for all v in $\mathfrak{g}$. The gradation on $\bigwedge(\mathfrak{g})$ induces a gradation on $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$ so that $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$ is a graded homology complex.

Lemma 4.1. *Denote by $C_\bullet(\mathfrak{g})$ the graded submodule $\bigwedge(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$ of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$.*

(i) *The graded module $C_\bullet(\mathfrak{g})$ is a graded subcomplex of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$.*

(ii) *The support of the homology of $C_\bullet(\mathfrak{g})$ is contained in $\mathcal{C}(\mathfrak{g})$.*

Proof. (i) Set:

$$\varepsilon := \wedge_{i=1}^{\ell} \varepsilon_i^{(0)} \wedge \cdots \wedge \varepsilon_i^{(d_i-1)}.$$

Then $C_\bullet(\mathfrak{g})$ is the ideal of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$ generated by ε since $\varepsilon_i^{(0)},\dots,\varepsilon_i^{(d_i-1)}$, $i = 1,\dots,\ell$ is a basis of $\mathrm{B}_\mathfrak{g}$ by Theorem 2.2,(i). According to Theorem 2.2,(iii), for $i = 1,\dots,\ell$ and for $m = 0,\dots,d_i-1$, $\varepsilon_i^{(m)}$ is a cycle of the complex $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$. Hence too is ε and $C_\bullet(\mathfrak{g})$ is a subcomplex of $\mathrm{S}(\mathfrak{g}\times\mathfrak{g}) \otimes_\Bbbk \bigwedge(\mathfrak{g})$ as an ideal generated by a cycle.

(ii) Let (x_0,y_0) be in $\mathfrak{g}\times\mathfrak{g} \setminus \mathcal{C}(\mathfrak{g})$ and let v be in $\mathfrak{g}$ such that $\langle v,[x_0,y_0]\rangle \neq 0$. For some affine open subset O of $\mathfrak{g}\times\mathfrak{g}$, containing (x_0,y_0), $\langle v,[x,y]\rangle \neq 0$ for all (x,y) in O. Then $\mathrm{d}v$ is an invertible element of $\Bbbk[O]$. For c a cycle of $\Bbbk[O] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} C_\bullet(\mathfrak{g})$,

$$\mathrm{d}(v \wedge c) = (\mathrm{d}v)c$$

so that c is a boundary of $\Bbbk[O] \otimes_{\mathrm{S}(\mathfrak{g}\times\mathfrak{g})} C_\bullet(\mathfrak{g})$. □

Theorem 4.2. (i) *The complex $C_\bullet(\mathfrak{g})$ has no homology in degree bigger than* $\mathrm{b}_\mathfrak{g}$.

(ii) *The ideal $I_\mathfrak{g}$ has projective dimension $2n-1$.*

(iii) *The algebra $\mathrm{S}(\mathfrak{g}\times\mathfrak{g})/I_\mathfrak{g}$ is Cohen-Macaulay.*

(iv) *The projective dimension of the module $\bigwedge^n(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_\mathfrak{g}}(\mathrm{B}_\mathfrak{g})$ equals n.*

Proof. (i) Let Z be the space of cycles of degree $\mathrm{b}_\mathfrak{g}+1$ of $C_\bullet(\mathfrak{g})$. Then we deduce from $C_\bullet(\mathfrak{g})$ the complex

$$0 \longrightarrow C_{2n+\ell}(\mathfrak{g}) \longrightarrow \cdots \longrightarrow C_{n+\ell+2}(\mathfrak{g}) \longrightarrow Z \longrightarrow 0.$$

According to Lemma 4.1,(ii), the support of its homology is contained in $\mathcal{C}_\mathfrak{g}$. In particular, its codimension in $\mathfrak{g}\times\mathfrak{g}$ is

$$4n + 2\ell - (2n+2\ell) = 2n = n + n - 1 + 1$$

According to Proposition 3.9, for $i = n + \ell + 2, \ldots, 2n + \ell$, $C_i(\mathfrak{g})$ has projective dimension at most n. Hence, by Corollary A.3, this complex is acyclic and Z has projective dimension at most $2n - 2$, whence the assertion.

(ii) and (iii) Since $\mathrm{B}_{\mathfrak{g}}$ is a free module of rank $\mathrm{b}_{\mathfrak{g}}$, $\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ is a free module of rank 1. By definition, the short sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}) \longrightarrow 0$$

is exact, whence the short exact sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \textstyle\bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \longrightarrow 0.$$

Moreover, by Proposition 3.9, $\mathfrak{g} \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ has projective dimension at most 1. Then, by (i), $I_{\mathfrak{g}}$ has projective dimension at most $2n - 1$. As a result the $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})$-module $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has projective dimension at most $2n$. Then by Auslander-Buchsbaum's theorem [Bou98, §3, n°3, Théorème 1], the depth of the graded $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})$-module $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is at least

$$4\mathrm{b}_{\mathfrak{g}} - 2\ell - 2n = 2\mathrm{b}_{\mathfrak{g}}$$

so that, according to [Bou98, §1, n°3, Proposition 4], the depth of the graded algebra $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is at least $2\mathrm{b}_{\mathfrak{g}}$. In other words, $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is Cohen-Macaulay since it has dimension $2\mathrm{b}_{\mathfrak{g}}$. Moreover, since the graded algebra $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has depth $2\mathrm{b}_{\mathfrak{g}}$, the graded $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})$-module $\mathrm{S}(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has projective dimension $2n$. Hence $I_{\mathfrak{g}}$ has projective dimension $2n - 1$.

(iv) By (i), $I_{\mathfrak{g}}$ has projective dimension $2n - 1$. Hence, according to Proposition 3.9 and according to (ii) and Corollary A.3, $\bigwedge^n(\mathfrak{g}) \wedge \bigwedge^{\mathrm{b}_{\mathfrak{g}}}(\mathrm{B}_{\mathfrak{g}})$ has projective dimension n. □

Theorem 4.3. *The subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is Cohen-Macaulay and normal. Furthermore, $I_{\mathfrak{g}}$ is a prime ideal.*

Proof. According to Theorem 4.2,(iii), the subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is Cohen-Macaulay. According to [Po08, Theorem 1], it is smooth in codimension 1. So by Serre's normality criterion [Bou98, §1, n°10, Théorème 4], it is normal. In particular, it is reduced and $I_{\mathfrak{g}}$ is radical. According to [Ri79], $\mathcal{C}(\mathfrak{g})$ is irreducible. Hence $I_{\mathfrak{g}}$ is a prime ideal. □

Appendix A. Projective dimension and cohomology

Recall in this section classical results. Let X be a Cohen-Macaulay irreducible affine algebraic variety and let S be a closed subset of codimension p of X. Let $P_\bullet$ be a complex of finite projective $\Bbbk[X]$-modules whose lenght l is finite and let ε be an augmentation morphism of $P_\bullet$ whose image is R, whence an augmented complex of $\Bbbk[X]$-modules,

$$0 \longrightarrow P_l \longrightarrow P_{l-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} R \longrightarrow 0.$$

Denote by $\mathcal{P}_\bullet$, $\mathcal{R}$, $\mathcal{K}_0$ the localizations on X of $P_\bullet$, R, the kernel of ε respectively and denote by $\mathcal{K}_i$ the kernel of the morphism $\mathcal{P}_i \longrightarrow \mathcal{P}_{i-1}$ for i positive integer.

Lemma A.1. *Suppose that S contains the support of the homology of the augmented complex $P_\bullet$.*

(i) *For all positive integer $i < p - 1$ and for all projective $\mathcal{O}_X$-module $\mathcal{P}$, $\mathrm{H}^i(X \setminus S, \mathcal{P})$ equals zero.*

(ii) *For all nonnegative integer $j \leq l$ and for all positive integer $i < p - j$, the cohomology group $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-j})$ equals zero.*

Proof. (i) Let $i < p-1$ be a positive integer. Since the functor $\mathrm{H}^i(X \setminus S, \bullet)$ commutes with the direct sum, it suffices to prove $\mathrm{H}^i(X \setminus S, \mathcal{O}_X) = 0$. Since S is a closed subset of X, one has the relative cohomology long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i_S(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^i(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{O}_X) \longrightarrow \mathrm{H}^{i+1}_S(X, \mathcal{O}_X) \longrightarrow \cdots .$$

Since X is affine, $\mathrm{H}^i(X, \mathcal{O}_X)$ equals zero and $\mathrm{H}^i(X \setminus S, \mathcal{O}_X)$ is isomorphic to $\mathrm{H}^{i+1}_S(X, \mathcal{O}_X)$. Since X is Cohen-Macaulay, the codimension p of S in X equals the depth of its ideal of definition in $\Bbbk[X]$ [MA86, Ch. 6, Theorem 17.4]. Hence, according to [Gro67, Theorem 3.8], $\mathrm{H}^{i+1}_S(X, \mathcal{O}_X)$ and $\mathrm{H}^i(X \setminus S, \mathcal{O}_X)$ equal zero since $i + 1 < p$.

(ii) Let j be a nonnegative integer. Since S contains the support of the homology of the complex $P_\bullet$, for all nonnegative integer j, one has the short exact sequence of $\mathcal{O}_{X\setminus S}$-modules

$$0 \longrightarrow \mathcal{K}_{j+1}\,|_{X\setminus S} \longrightarrow \mathcal{P}_{j+1}\,|_{X\setminus S} \longrightarrow \mathcal{K}_j\,|_{X\setminus S} \longrightarrow 0$$

whence the long exact sequence of cohomology

$$\cdots \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^i(X \setminus S, \mathcal{K}_j) \longrightarrow \mathrm{H}^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^{i+1}(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \cdots .$$

Then, by (i), for $0 < i < p-2$, the cohomology groups $\mathrm{H}^i(X \setminus S, \mathcal{K}_j)$ and $\mathrm{H}^{i+1}(X \setminus S, \mathcal{K}_{j+1})$ are isomorphic since P_{j+1} is a projective module. Since $\mathcal{P}_i = 0$ for $i > l$, $\mathcal{K}_{l-1}$ and $\mathcal{P}_l$ have isomorphic restrictions to $X \setminus S$. In particular, by (i), for $0 < i < p-1$, $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-1})$ equal zero. Then, by induction on j, for $0 < i < p - j$, $\mathrm{H}^i(X \setminus S, \mathcal{K}_{l-j})$ equals zero. □

Proposition A.2. *Let R' be a $\Bbbk[X]$-module containing R. Suppose that the following conditions are verified:*

(1) *p is at least $l + 2$,*
(2) *X is normal,*
(3) *S contains the support of the homology of the augmented complex $P_\bullet$.*

(i) *The complex $P_\bullet$ is a projective resolution of R of length l.*
(ii) *Suppose that R' is torsion free and that S contains the support in X of R'/R. Then $R' = R$.*

Proof. (i) Let j be a positive integer. One has to prove that $\mathrm{H}^0(X, \mathcal{K}_j)$ is the image of P_{j+1}. By Condition (3), the short sequence of $\mathcal{O}_{X\setminus S}$-modules

$$0 \longrightarrow \mathcal{K}_{j+1}\big|_{X\setminus S} \longrightarrow \mathcal{P}_{j+1}\,|_{X\setminus S} \longrightarrow \mathcal{K}_j\big|_{X\setminus S} \longrightarrow 0$$

is exact, whence the cohomology long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_j) \longrightarrow \mathrm{H}^1(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \cdots .$$

By Lemma A.1,(ii), $\mathrm{H}^1(X \setminus S, \mathcal{K}_{j+1})$ equals 0 since $1 < p - l + j + 1$, whence the short exact sequence

$$0 \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_j) \longrightarrow 0.$$

Since the codimension of S in X is at least 2 and since X is irreducible and normal, the restriction morphism from P_{j+1} to $\mathrm{H}^0(X \setminus S, \mathcal{P}_{j+1})$ is an isomorphism. Let φ be in $\mathrm{H}^0(X, \mathcal{K}_j)$. Then there exists an element

ψ of P_{j+1} whose image ψ' in $\mathrm{H}^0(X, \mathcal{K}_j)$ has the same restriction to $X \setminus S$ as φ. Since P_j is a projective module and since X is irreducible, P_j is torsion free. Then $\varphi = \psi'$ since $\varphi - \psi'$ is a torsion element of P_j, whence the assertion.

(ii) Let $\mathcal{R}'$ be the localization of R' on X. Arguing as in (i), since S contains the support of R'/R and since $1 < p - l$, the short sequence

$$0 \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{K}_0) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{P}_0) \longrightarrow \mathrm{H}^0(X \setminus S, \mathcal{R}') \longrightarrow 0$$

is exact. Moreover, the restriction morphism from P_0 to $\mathrm{H}^0(X/S, \mathcal{P}_0)$ is an isomorphism since the codimension of S in X is at least 2 and since X is irreductible and normal. Let φ be in R'. Then for some ψ in P_0, $\varphi - \varepsilon(\psi)$ is a torsion element of R'. So $\varphi = \varepsilon(\psi)$ since R' is torsion free, whence the assertion. □

Corollary A.3. *Let $C_\bullet$ be a homology complex of finite $\Bbbk[X]$-modules whose length l is finite and positive. For $j = 0, \ldots, l$, denote by Z_j the space of cycles of degree j of $C_\bullet$. Suppose that the following conditions are verified:*

(1) *S contains the support of the homology of the complex $C_\bullet$,*
(2) *for all i, C_i is a submodule of a free module,*
(3) *for $i = 1, \ldots, l$, C_i has projective dimension at most d,*
(4) *X is normal and $l + d \leq p - 1$.*

Then $C_\bullet$ is acyclic and for $j = 0, \ldots, l$, Z_j has projective dimension at most $l + d - j - 1$.

Proof. Prove by induction on $l - j$ that the complex

$$0 \longrightarrow C_l \longrightarrow \cdots \longrightarrow C_{j+1} \longrightarrow Z_j \longrightarrow 0$$

is acyclic and that Z_j has projective dimension at most $l + d - j - 1$. For $j = l$, Z_j equals zero since C_l is torsion free by Condition (2) and since Z_l a submodule of C_l, supported by S by Condition (1). Suppose $j \leq l - 1$ and suppose the statement true for $j + 1$. By Condition (3), C_{j+1} has a projective resolution $P_\bullet$ whose length is at most d and whose terms are finitely generated. By induction hypothesis, Z_{j+1} has a projective resolution $Q_\bullet$ whose length is at most $l + d - j - 2$ and whose terms are finitely generated, whence an augmented complex $R_\bullet$ of projective modules whose length is $l + d - j - 1$,

$$0 \longrightarrow Q_{l+d-j-2} \oplus P_{l+d-j-1} \longrightarrow \cdots \longrightarrow Q_0 \oplus P_1 \longrightarrow P_0 \longrightarrow Z_j \longrightarrow 0.$$

Denoting by d the differentials of $Q_\bullet$ and $P_\bullet$, the restriction to $Q_i \oplus P_{i+1}$ of the differential of $R_\bullet$ is the map

$$(x, y) \mapsto (\mathrm{d}x, \mathrm{d}y + (-1)^i \delta(x)),$$

with δ the map which results from the injection of Z_{j+1} into C_{j+1}. Since $P_\bullet$ and $Q_\bullet$ are projective resolutions, the complex $R_\bullet$ is a complex of projective modules having no homology in positive degree. Hence the support of the homology of the augmented complex $R_\bullet$ is contained in S by Condition (1). Then, by Proposition A.2 and Condition (4), $R_\bullet$ is a projective resolution of Z_j of length $l + d - j - 1$ since Z_j is a submodule of a free module by Condition (2), whence the corollary since $Z_0 = C_0$ by definition. □

Remark A.4. Let $\mathrm{D}(X)$ be the bounded derived category of finite $\Bbbk[X]$-modules. For E an object of $\mathrm{D}(X)$, denote by $\mathrm{Supp}(E)$ the union of the supports in X of the homology modules $\mathrm{H}_i(E)$ of E. By definition, the *homological dimension* of E, written $\mathrm{hd}\,(E)$, is the smallest integer s such that E is quasi-isomorphic to a complex of projective $\Bbbk[X]$-modules of length s. If no such integer exists, $\mathrm{hd}\,(E) = \infty$. Since X is Cohen-Macaulay, according to [MA86, Ch. 6, Theorem 17.4], we have the following proposition:

Proposition A.5. [BM02, Corollary 5.5] *Let E be a non trivial object of* $\mathrm{D}(X)$*. Then for all irreducible component* Γ *of* $\mathrm{Supp}(E)$*,*

$$\dim X - \dim \Gamma \leq \mathrm{hd}\,(E).$$

Corollary A.3 is a little bit smilar to Proposition A.5. But it is not a consequence of Proposition A.5 since its proof does not use the normality of X.

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